ON THE MOTION OF A RIGID BODY ABOUT A STATIONARY POINT FIXED ON THE EARTH'S SURFACE

(O DVIZHENII TVERDOGO TELA VOKRUG NEPODVIZHNOI TOCHKI, ZAKREPLENNOI NA POVERKHNOSTI ZEMLI)

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As we know, the classical Euler-Poisson equations describing the motion of a heavy rigid body about a stationary point are approximate in the case of coordinate systems rigidly fixed to the earth.

In order to derive the exact equations one must take into account the Coriolis inertial forces and replace the parallel gravitational force field with a Newtonian central force field.

Taking as our coordinate system $Ox_1y_1z_1$ fixed to the earth the geocentric system (see figure) whose origin O is a point on the earth's surface coinciding with the origin of the system $Oxyz$ rigidly connected to the solid, we obtain (on the basis of [1]) the following exact equations of motion:

$$A\frac{dp^*}{d\tau} + (C-B)(q^*r^* + \omega^2\beta'\beta'') + \omega A(\beta' r^* - \beta'' q^*) + \omega(C-B)(q^*\beta'' + r^*\beta') = n_1$$

$$n_1 = \gamma''\frac{\partial u}{\partial \gamma'} - \gamma'\frac{\partial u}{\partial \gamma''} \tag{1}$$

$$\frac{d\gamma}{d\tau} = r^*\gamma' - q^*\gamma'', \qquad \frac{d\beta}{d\tau} = r^*\beta' - q^*\beta'' \qquad (ABC,\ 123,\ \gamma\gamma'\gamma'',\ \beta\beta'\beta'',\ p^*q^*r^*)$$

Here $u = u(\gamma, \gamma', \gamma'')$ is the force function of the Newtonian force field; ω is the vector of the earth's angular velocity; β, β', β'' and $\gamma, \gamma', \gamma''$ are the direction cosines of ω and of the axis z_1 in the coordinate system $Oxyz$; p^*, q^* and r^* are the components of the angular velocity of the solid.

Let us make use of the approximate relations of [2] to express n_i $(i = 1, 2, 3)$,

$$n_1 = -Mg(y_0\gamma'' - z_0\gamma') + 3gR^{-1}(C-B)\gamma'\gamma'' \qquad (123,\ x_0y_0z_0,\ ABC,\ \gamma\gamma'\gamma'')$$

Here g is the acceleration due to gravity on the earth's surface. These expressions are valid provided the dimensions of the solid are small as compared with the earth's radius R. Making the substitutions

$$p^* = \alpha p - \omega\beta, \quad q^* = \alpha q - \omega\beta', \quad r^* = \alpha r - \omega\beta'', \quad t = \alpha\tau, \quad \alpha^2 = 3gR^{-1}$$

we rewrite Equations (1) in dimensionless variables,

$$A\frac{dp}{dt}+(C-B)\,qr=-(y_1\gamma''-z_1\gamma')+(C-B)\,\gamma'\gamma''$$

$$\frac{d\gamma}{dt}=r\gamma'-q\gamma''-\lambda\,(\beta''\gamma'-\beta'\gamma''),\qquad \frac{d\beta}{dt}=r\beta'-q\beta''\qquad (ABC,\ x_1y_1z_1,\ \gamma\gamma'\gamma'',\ \beta\beta'\beta'') \tag{2}$$

$$x_1={}^1/_3 Mx_0R,\qquad y_1={}^1/_3 My_0R,\qquad z_1={}^1/_3 Mz_0R,\qquad \lambda=\omega\,/\,\alpha$$

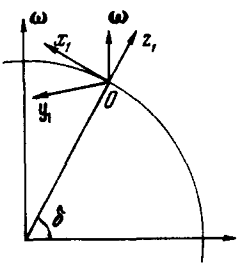
The resulting nine equations (2) have three geometric integrals

$$\beta^2+\beta'^2+\beta''^2=1,\qquad \gamma\beta+\gamma'\beta'+\gamma''\beta''=\sin\delta \tag{3}$$

$$\gamma^2+\gamma'^2+\gamma''^2=1 \tag{4}$$

and the generalized energy integral

$$ {}^1/_2\,[A\,(p^2+\gamma^2)+B\,(q^2+\gamma'^2)+C\,(r^2+\gamma''^2)]+x_1\gamma+ \\ +y_1\gamma'+z_1\gamma''-\lambda\,(Ap\beta+Bq\beta'+Cr\beta'')=h \tag{5}$$



Here δ is the latitude of the fixed point.

Thus, this problem can be regarded as a further generalization of the familiar problem concerning the motion of a rigid body about a stationary point in a Newtonian force field.

We note that the equations of the problem in form (2) can be obtained immediately by considering the motion of the rigid body relative to a coordinate system in translational motion together with the fixed point on the earth's surface.

We shall reduce system (2) to seven equations.

To do this we express the quantities β, β' and β'' with the aid of the Euler angles θ, φ, and ψ in terms of

$$\gamma,\ \gamma',\ \gamma'',\ \psi$$

$$\beta=\cos\delta\,(\cos\varphi\,\cos\psi-\cos\theta\,\sin\varphi\,\sin\psi)+\sin\delta\,\sin\theta\,\sin\varphi$$

$$\beta'=-\cos\delta\,(\sin\varphi\,\cos\psi+\cos\varphi\,\cos\theta\,\sin\psi)+\sin\delta\,\cos\varphi\,\sin\theta \tag{6}$$

$$\beta''=\cos\delta\,\sin\theta\,\sin\psi+\sin\delta\cos\theta,\qquad \tan\varphi=\gamma\,/\,\gamma',\qquad \cos\theta=\gamma''$$

We can then determine the quantities p, q, r, γ, γ', γ'' and ψ from the first six equations of system (2) after making substitution (6) and from the equation

$$d\psi\,/\,dt=(p\gamma+q\gamma')\,/\,(\gamma^2+\gamma'^2) \tag{7}$$

The resulting system of seven equations has the integral (4) and integral (5), incorporating substitution (6).

Some particular solutions can be pointed out for Equations (2). For example, under the condition $A=B$, $x_1=0$ and $y_1=0$ the following particular solution exists

$$p=k\gamma,\ q=k\gamma',\ r=r_0,\ \gamma'=\gamma_0'',\ \beta=\rho\gamma,\ \beta'=\rho\gamma',\ \beta''=\beta_0'' \tag{8}$$

The constants k, ρ, r_0, γ_0'' and β_0'' in this solution are connected by the algebraic expressions

$$(k - \lambda\rho)(\gamma_0'' \rho - \beta_0'') = 0, \qquad \rho = (\sin\delta - \gamma_0''\beta_0'') / (1 - \gamma_0''^2)$$
$$k^2\gamma_0'' + \left[-\frac{Cr_0}{A} + \lambda(\beta_0'' - \rho\gamma_0'')\right]k + \frac{z_1}{A} - \frac{(A - C)\,\gamma_0''}{A} = 0 \tag{9}$$

The quantities γ and γ' in relations (8) can be determined from the solution of the corresponding harmonic oscillation equation.

Relative to the earth, solution (8) corresponds to the uniform rotation of a rigid body about a fixed axis.

Naturally, owing to the complexity of the above problem, the number of cases in which Equations (2) are reducible to quadratures is smaller than in the problem concerning the motion of a rigid body about a stationary point in a Newtonian force field.

Let us estimate the parameter λ: we find it equal to $\lambda = 0.03$ for the earth and $\lambda = 0.001$ for the moon. Thus, in order to find the solutions of the problem in question, it is possible to make use of the small parameter method of Poincaré. As we know, to construct a solution in the form of a series in powers of the small parameter λ it is sufficient to find the general, and in some cases even a particular solution of the simplified (for $\lambda = 0$) system of initial equations.

In this case Equations (2) become the familiar equations of motion of a rigid body about a stationary point in a Newtonian force field and relate only the quantities p, q, r, γ, γ' and γ''.

The remaining equation (7) is immediately integrable with known p, r, γ and γ'.

Hence, the use of the small parameter method for solving the problem of motion of a rigid body about a stationary point fixed on the earth's surface depends entirely on the solution of the problem about the motion of a rigid body about a stationary point in a Newtonian force field.

As regards the latter problem, which has already been studied in considerable detail, general solutions have been found in the two cases

$$1)\quad x_0 = 0, \quad y_0 = 0, \quad z_0 = 0; \qquad 2)\quad A = B, \quad x_0 = y_0 = 0$$

Furthermore, a number of simple particular solutions is given in [3-8] for the first of the above cases.

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